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ON STABILITY UNDER CONSTANTLY ACTING PERTURBATIONS*

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A general formulation of the Volterra theorem on the parametric stability of steady motion of a gyrostat (/1/ page 261) (**) is presented. The fundamental aspects of its proof together with the concept of Liapunov's direct method are extended to more general theorems on parametric stability. Volterra's geometric constructions are generalized so that his proofs can be applied to certain theorems on stability under constantly acting perturbations, and that these and theorems on parametric stability can be considered from a single point of view. Some applications of Volterra's theorem are examined, and a few examples presented.

1. We use the following notation: x for the vector of an n-dimensional real phase space R^n with zero at θ and X for the region of that space containing inside it point θ; p for the vector of an m-dimensional space of parameters P with zero at 0; T for the real positive semi-axis of time; $O^\varepsilon(\theta)$ for the ε-neighborhood of point θ; $O_\cdot^\varepsilon(\theta)$ for the same neighborhood with point θ removed; $O_\delta^\varepsilon(\theta)(\delta < \varepsilon)$ for the closing of the region obtained by excluding the neighborhood $O^\delta(\theta)$ from neighborhood $O^\varepsilon(0)$ (the "annular region" G shown shaded in Fig.1,a); $\Gamma(M)$, M^*, and $O^\varepsilon(M)$ for the boundary and closure, and the ε-neighborhood of some set $M \subset X$; TM $(M \subset X)$ for the set of points (t, x) in the space $T \times R^n$ such that $t \in T$ and $x \in M$, for instance, region $TO_\delta^\varepsilon(0)$ (shown shaded in Fig.1,b).

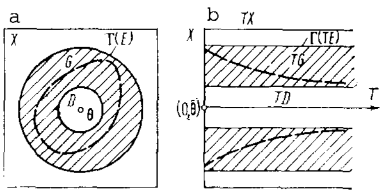

Fig.1

Let $V(x)$ be a continuous function and $K(\lambda) \subset X$ a component of a set of the level of $V(x) = \lambda$. We call $K(\lambda)$ the centrally separating component in X, if it is bounded and separates some region $U(\lambda) \ni \theta$ from the exterior of that region. Such components exist, for example, in a positive definite function. The boundary of region $U(\lambda)$ is an $(n-1)$-dimensional cycle /2/ that evidently belongs to $K(\lambda)$. If function V depends in addition to x on parameter p, it will be denoted by $V_p(x)$ and, respectively $K_p(\lambda)$ and $U_p(\lambda)$. The component $K_p(\lambda)$ is shown in Fig.1,a by dash lines.

Let us consider the autonomous dynamic system

$$x^\cdot = f_p(x) \quad (f_0(\theta) = 0) \tag{1.1}$$

where $f_p(x)$ is an n-dimensional vector function continuous with respect to $x \in X$ and $p \in P$.

The conditions under which the equilibrium position θ of the unperturbed system $(p = 0)$ is stable in the sense that for each $\varepsilon > 0$ there exist such $\delta > 0$ and $\eta > 0$ that

$$(\forall t \geqslant t_0) \quad x_p(t_0, x_0; t) \in O^\varepsilon(\theta) \tag{1.2}$$

is valid for every $x_0 \in O^\delta(\theta)$ and all p, $\|p\| < \eta$, were investigated in /1/. It means that for $t > t_0$ the integral curve whose initial point is (t_0, x_0) is contained in ε-tube $TO^\varepsilon(\theta)$ (p is fixed).

Owing to the stationarity of system (1.1) at the beginning, as well as at the end, as defined above, it is possible to introduce the condition for "each $t_0 \in T$." We obtain two equivalent statements

$$(\forall t_0 \in T)(\forall \varepsilon > 0)(\exists \delta, \ \eta > 0)(\forall x_0 \in O^\delta(\theta))(\forall \|p\| < \eta): \mathrm{A} \tag{1.3}$$

$$(\forall \varepsilon > 0)(\exists \delta, \ \eta)(\forall x_0 \in O^\delta(\theta))(\forall \|p\| < \eta)(\forall t_0 \in T): \mathrm{A} \tag{1.4}$$

where A denotes the expression (1.2).

These statements have also a meaning for the nonautonomous system

$$x^\cdot = f_p(t, x) \quad (f_0(t, \theta) \equiv 0) \tag{1.5}$$

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**) V. V. Rumiantsev had repeatedly pointed out Volterra's foresight in anticipating in /1/ a number of later developments in the theories of the gyrostat and of stability. His theorem and its general character were brought to the writer's attention by V. N. Rubanovskii.

but in this case they are not equivalent: in (1.3) the numbers δ and η depend on ε and t_0, while in (1.4) only on ε (parametric stability with respect to $t_0 \in T$.

We thus obtain the following definition of Volterra's concept of parametric stability.

Definition 1.1. The equilibrium position θ of the unperturbed system $x^{\cdot} = f_0(x)$ or $x^{\cdot} = f_0(t, x)$ is called stable (stable uniformly with respect to t_0) under perturbation of p, if the statement (1.3) or (1.4) is valid.

This definition may be used for perturbations of a more general type such as constantly acting perturbations, if p is assumed to be the vector function $p(t, x) = [p^1(t, x), \ldots, p^n(t, x)]$ and P to be the space of such functions (perturbations) with one or another metric. If functions $p(t, x)$ are continuous and bounded in TX, then, by writing system (1.5) in the form

$$x^{\cdot} = f_0(t, x) + p(t, x) \quad (f_0(t, \theta) \equiv 0) \tag{1.6}$$

and introducing the norm

$$\| p \| = \sup | p^i(t, x) |, \quad (t, x) \in TX, \quad i = 1, \ldots, n \tag{1.7}$$

we obtain the definition /3/

Definition 1.2. The equilibrium position θ of the unperturbed system $x^{\cdot} = f_0(t, x)$ is called stable (uniformly stable with respect to t_0)) under constantly acting perturbations, if the statement (1.3) or (1.4) is valid.

Remark 1.1. Perturbations of the right-hand sides of differential equations were considered by Liapunov in implicit form in his first method (see Chetaev's comments in /4/), and earlier, by Poincaré /5/. (Ch.18). A related question was considered in /6/. The notion that it is necessary to take into account constantly acting perturbations was explicitly suggested by Chetaev /7/. The introduced by him concept of constantly acting perturbing forces, which is at the base of the formulation of the known stability postulate /7/, was used in /7, 8/ for autonomous perturbations. A class of potential perturbations was defined in /7/.

The definition of stability under constantly acting perturbations given above, appeared in /3/. Definitions equivalent to it in essential features appeared in /9,10/ (in /9/ parameter ε is taken as the perturbation norm, and in /10/ stability was considered with respect to functions of the type φ_i in /4/ in /7/ with respect to "observed functions" Φ_k and F_k). The considered concept of stability was extended in /11,12/ to the case of metric in the perturbation space in the mean, and to stability with respect to probability.

The concept of the limit point of the stability region /13,14/ represents a particular case of parametric stability definition 1.1 for a special type of parameters. Related problems were considered in /15—17/, and a more general concept of stability under parametric perturbations appeared in /18,19/.

2. In the defined above notation the Volterra theorem may be formulated as follows.

Theorem 2.1. If system (1.1) has for every p the first integral $V_p(x)$, function $V_p(x)$ is continuous with respect to x and p, and function $V_0(x)$ positive definite at point θ, the equilibrium position θ of the unperturbed system is parametrically stable in the meaning of definition 1.1.

The proof in /1/ is, except the terminology, as follows.

Let us assume that the closed region X is sufficiently small for being contained in the region of positive definiteness of function $V_0(x)$, and that ε is any positive number. We shall consider the neighborhood $E = O^{\varepsilon}(\theta) \subset X$ (see Fig.1,a). Since function $V_0(x)$ is continuous, there must exist a neighborhood $D = O^{\delta}(\theta) \subset E$ such that

$$\sup_{x \in D^*} V_0(x) < \inf_{x \in \Gamma(E)} V_0(x) \tag{2.1}$$

Owing to the closure of region X, function V is uniformly continuous with respect to x and p, respectively, in that region, i.e.

$$(\forall \varepsilon_1 > 0)(\exists \eta > 0)(\forall \| p \| < \eta)(\forall x \in X): , \quad | V_p(x) - V_0(x) | < \varepsilon_1 \tag{2.2}$$

which implies that inequality (2.1) is coarse with respect to p, hence there exists an $\eta > 0$ such that (2.1) remains valid for $\| p \| < \eta$. To prove this $\varepsilon_1 = (\lambda'' - \lambda')/2$, where λ' and λ'' are the left- and right-hand sides of (2.1), was set in /1/.

Since function $V_p(x)$ does not increase along the trajectory of system (1.1) and inequality (2.1) holds for $\| p \| < \eta$, hence any trajectory of system (1.1$'$) that had intersected $\Gamma(D)$ cannot intersect $\Gamma(E)$, i.e. G is a trap for trajectories that begin in D. This proves the theorem.

Thus the conclusion of parametric stability is derived in /1/ on the basis of the following three conditions which we shall call the Volterra conditions.

1°. In any neighborhood $O^\varepsilon(\theta)$ there exists region $G = O_\delta^\varepsilon(\theta)$ for which (2.1) is satisfied;

2°. The property of region G defined by (2.1) is coarse with respect to p;

3°. In any annular region $G \subset X$ function V_p does not increase along trajectories when $\|p\|$ is fairly small.

Remark 2.1. The theorem formulation in /1/ contains the condition of isolation of the equilibrium position θ for $p = 0$. Since that condition was used there only for proving the positive definiteness of function $V_0(x)$ for the considered mechanical system, and because it is not used in the proof of parametric stability, hence it has been omitted in the above formulation.

Remark 2.2. If region G satisfies (2.1), it contains at least one centrally dividing component of function $V_0(x)$. Hence condition 1° implies the existence of a "concentric" set $S_0 = \{K_0(\lambda)\}$ of centrally dividing components that converge at point θ as $\lambda \to 0$, and satisfy the monotonicity condition

$$\lambda_1 < \lambda_2 \Rightarrow U_0(\lambda_1) \subset U_0(\lambda_2) \tag{2.3}$$

When the equilibrium position θ is isolated, that set is regular and fills some neighborhood $Q_{.}(\theta)$ (with point θ removed), i.e. it constitutes the topographic system of Poincaré. A function containing such set of components will be called positive regular.

Isolation of the equilibrium position and the positive definiteness of function $V_0(x)$ is not indispensable either for the existence of set S_0 which satisfies (2.3), or for the positive regularity of that function.

3. The concept of the Liapunov second method provides the possibility of extending Theorem 2.1 to the case in which the Liapunov function plays the part of function V. For this it is necessary to replace the condition that $V_p(x)$ must be the first integral by a condition that is sufficient for satisfying the Volterra condition 3°, or by that condition itself.

In terms of Liapunov function condition 3° is of the form

$$(\forall G \subset X)(\exists \eta > 0)(\forall \|p\| < \eta)(\forall x \in G):\ V_p^{\cdot}(x) \leqslant 0 \tag{3.1}$$

where $V_p^{\cdot}(x)$ is the derivative defined by system (1.1), and $G = O_\delta^\varepsilon(\theta)$. We obviously obtain a generalization of Theorem 2.1.

Theorem 3.1. If function $V_p(x)$ satisfies the following conditions: $V_p(x)$ is smooth with respect to x and continuous with respect to p, $V_0(x)$ is positive definite at point θ, and the derivative of function $V_p(x)$ satisfies (3.1) by virtue of system (1.1), then the equilibrium position of the unperturbed system is parametrically stable.

Let us consider the case when the Liapunov function $V_0(x)$ is specified only for the unperturbed system. Setting $V_p(x) \equiv V_0(x)$ we obtain the stationary with respect to p set of components $K_p(\lambda)$ such that condition 2° is trivially satisfied. Although then $V_p^{\cdot}(x) \not\equiv V_0^{\cdot}(x)$, the negative definiteness of $V_0^{\cdot}(x)$ is sufficient for (3.1). Although the inequality $V_0^{\cdot}(x) < 0$ may, possibly, be violated in $O^\varepsilon_{.}(\theta)$ when $p \neq 0$, it remains valid in any closed region $O_\delta^\varepsilon(\theta)$ if $\|p\|$ is fairly small.

Thus we obtain the theorem /13,14/.

Theorem 3.2. If there exists a smooth function $V_0(x)$ that is positive definite at point θ and has a negative definite derivative by virtue of the unperturbed system $x^{\cdot} = f_0(x)$, the equilibrium position of that system is parametrically stable.

Since an asymptotic stability with respect to x_0 and t_0 of the equilibrium position is necessary sufficient for the existence of the Liapunov functions appearing in Theorem 3.2 (see /19/), hence such stability of the unperturbed system equilibrium position is sufficient for its parametric stability.

Theorem 3.2 was proved in /13,14/ in connection with the problem of safety of the stability region boundary. The point of parameter space is called in /13/ safe, if for respective parameter values the investigated equilibrium position is parametrically stable in the meaning of definition (1.1). Coefficients of the characteristic equation of the linearized system were taken in /13/ as parameters, and the stability region was understood to be the Routh—Hurwitz region of the parameter space. The simplest noncoarse systems (one zero or two imaginary roots) were analyzed in /13/, but in constructions required for proving the considered theorem only the existence of the Liapunov function with negative definite derivative by virtue of the unperturbed system was used. These constructions were carried out in a general form in /14/.

It was shown in /17/ (the basic theorem) that when the Liapunov function appearing in Theorem 3.2 exists for $p = 0$, then, when $p \neq 0$ there exists the closed region $A \ni \theta$

which is an invariant set of the perturbed system whose diameter approaches zero as $p \to 0$. It follows from the above that a region bounded by any centrally dividing component $K_0(\lambda)$ may be taken as region A, since any arbitrarily small component $K_0(\lambda)$ an fairly small $\|p\|$ lies in the region $V^{\cdot} < 0$.

4. Let us consider the extension of the Volterra theorem to the case of constantly acting perturbations for system (1.6) (see definition 1.2). Let P be the space of perturbations $p(t, x)$ with norm (1.7), To extend the theorems in Sect.3 to this case we, first, establish the analogy with constructions in Sects. 2 and 3. Point $x \in X$ of constructions in Sects.2 and 3 corresponds to point $(t, x) \in TX$, function $V_p(t, x)$ to function $V_p(x)$, region $TO^\varepsilon(\theta)$ in TX corresponds to neighborhood $O^\varepsilon(\theta)$ in X, and the cylindrical regions TX, $TE = TO^\varepsilon(\theta)$, $TD = TO^\delta(\theta)$, and $TG = TO^{\varepsilon}{}_\delta(\theta)$ to regions X, E, D, and G (see Fig.1,b). The component $K_p(\lambda) \subset TX$ now denotes a component of the set of level $V_p(t, x) = \lambda$, and region $U_p(\lambda) \subset TX$ obtains a similar meaning.

Now, repeating the reasoning of Sect..2, that the Volterra conditions $1^\circ - 3^\circ$ imply stability under continuously acting perturbations in the meaning of (1.4), i.e. stability uniform with respect to t_0.

Condition 1°. is satisfied for any positive definite function $V_0(t, x)$ which admits an infinitely small upper bound /2/. This follows directly from the definition of these concepts (with at least one centrally dividing component $K_0(\lambda)$ in TG, see Fig.1,b).

Condition 2° follows, as previously, from formula (2.2) which in conformity with the accepted here analogy, implies the continuity of $V_p(t, x)$ with respect to p that is uniform with respect to $(t, x) \in TX$, i.e.

$$(\forall \varepsilon_1 > 0)(\exists \eta > 0)(\forall \|p\| < \eta)(\forall (t, x) \in TX): \quad | V_p(t, x) - V_0(t, x) | < \varepsilon_1 \tag{4.1}$$

Condition 3° of the form (3.1) now becomes

$$(\forall TG = TO^{\varepsilon}{}_\delta(\theta))(\exists \eta > 0)(\forall \|p\| < \eta)(\forall (t, x) \in TG): \quad V_p^{\cdot}(t, x) \leqslant 0 \tag{4.2}$$

Thus Theorem 3.1 converts to the following theorem /20/.

Theorem 4.1. Let function $V_p(t, x)$ smooth with respect to (t, x) satisfy the following conditions: 1) $V_0(t, x)$ is positive definite and admits an infinitely small upper bound; 2) $V_p(t, x)$ is continuous with respect to p uniformly with respect to $(t, x) \in TX$, i.e. it satisfies (4.1), and 3) $V_p^{\cdot}(t, x)$ satisfies (4.2). Then the equilibrium position θ of the unperturbed system is stable under continuously acting perturbations uniformly with respect to t_0, i.e. (1.4) is satisfied.

When $p \neq 0$ function $V_p(t, x)$ corresponds to function $V(t, x)$ in /20/ which depends on t and x, and on perturbations $R(t, x)$. If one assumes that condition (4.2) is satisfied throughout region TE (e.g., when $V_p(t, x)$ is the first integral), then, as noted in /20/, the stipulation of the existence of an infinitely small upper bound can be disregarded. Note that then the stability is generally nonuniform with respect to t_0, i.e. (1.3) and not (1.4), is satisfied (in that case it is sufficient for the δ-tube TD from which initial points are taken to be imbedded in region $U(\lambda)$ not for all $t \in T$, but only on some segment $[0, t^*]$).

Let us now consider the case when the Liapunov function $V_0(t, x)$ is specified only for the unperturbed system. As in Sect.3 we set $V_p(t, x) \equiv V_0(t, x)$. Components $K_p(\lambda)$ are then stationary with respect to p, so that (4.1) is trivially satisfied. The derivative of function $V_0(t, x)$ is by virtue of system (1.6) of the form

$$V_p^{\cdot}(t, x) = V_0^{\cdot}(t, x) + (\operatorname{grad}_x V_0(t, x), p(t, x)) \tag{4.3}$$

where the parentheses in the last term denote the scalar product, and $V_0^{\cdot}(t, x)$ is the derivative by virtue of the unperturbed system $x^{\cdot} = f_0(t, x)$. As is evident from (4.3), the negative definiteness of function $V_0^{\cdot}(t, x)$ is insufficient for (4.2), but (4.2) will be satisfied when the supplementary condition of boundedness of $\operatorname{grad}_x V_0(t, x)$ in TX is satisfied /3/. Moreover, the last condition is sufficient for the existence of an infinitely small upper bound of function $V_0(t, x)$, which means that condition 1° is satisfied.

Theorem 4.2. If there exists a positive definite smooth function $V_0(t, x)$ whose derivative is by virtue of the unperturbed system $x^{\cdot} = f_0(t, x)$ negative definite and partial derivatives with respect to x are bounded in region TX, then the equilibrium position of the unperturbed system is stable under constantly acting perturbations uniformly with respect to t_0, i.e. (1.4) is satisfied.

Remark 4.1. The distinctive feature of Sect.4 in comparison with Sects.2 and 3 is the noncompactness of region TX and of components $K_p(\lambda)$. Owing to this the existence of an infinitely small upper bound of function $V_0(t, x)$ is necessary if condition 1° is to be satisfied. Hence conditions (4.1) and (4.2) do not follow, as previously, from the continuity of $V_p(t, x)$

with respect to p and from the negative definiteness of $V_0^{\cdot}(t, x)$, respectively. In the case of autonomous constantly acting perturbations in an autonomous system the above peculiarity is absent, and this case does not differ (as regards the considered above questions) from that of parametric perturbations of a perturbed autonomous system.

5. The theorems of Sects.2 and 3 can be somewhat amplified.

Theorem 5.1. If the conditions of Theorem 2.1 or 3.1 are satisfied, there exists for every neighborhood $E = O^{\varepsilon}(\theta)$ an $\eta > 0$ such that for any $\|p\| < \eta$ that neighborhood contains a stable invariant set $B \ni \theta$ (dependent on p) of system (1.1).

Proof. The conditions of Theorems 2.1 and 3.1 imply the existence of the concentric set $S_0 = \{K_0(\lambda)\}$ of centrally dividing components of function $V_0(x)$, which satisfies (2.3) (see Remark 2.2). That set is generally irregular (if it is regular, the theorem is obviously valid). It can be shown on the basis of /21/ that, when the set S_0 exists, another concentric set can be found whose component K_0 has the following properties: in any neighborhood $O(K_0)$ there exists a centrally separating component which contains in the region bounded by it the component K_0 and is of a high level than K_0 (nonseparating components of lower level may also exist in the neighborhood $O(K_0)$). We assume that set S_0 is chosen exactly so. Then the described property implies that: 1) the region bounded by any of the components of set S_0 is a stable invariant set of the unperturbed system; 2) the closed subset $S_0^* \subset S_0$ enclosed between the components K_0' and K_0'' ($K_0' \subset D$; $K_0'' \cap D = \varnothing$) is coarse with respect to p, however small the neighborhood $D = O^{\delta}(\theta)$, i.e. for every $\sigma > 0$ there exists an $\eta > 0$ (one and the same for all elements of set S_0^*) such that for $\|p\| < \eta$ a centrally separating component K_p of function $V_p(x)$ can be found in the neighborhood $O^{\sigma}(K_0)$ ($K_0 \subset S_0^*$), and that the set $S_p^* = \{K_p\}$ has the same property as S_0^*. The number η can be selected so small that $K_p' \subset D$, and the Volterra property 3° is satisfied in region $G = O_{\delta}^{\varepsilon}(\theta)$ (Fig.2).

Thus the set S_0 can disintegrate when $p \neq 0$ only inside the D-neighborhood, while the subset of components not contained in D remains topologically unchanged for all fairly small $\|p\|$. The region bounded by any component K_p^* of set S_p^* contained in G can be taken as the sought stable invariant set (Fig.2; point θ may in this case represent an unstable equilibrium position, or altogether not relate to the equilibrium position). Theorem 5.1 is proved.

Under conditions of Theorem 3.2 the set S_0 is regular (owing to the negative definiteness of function $V_0^{\cdot}(x)$). From this follows the following theorem (which appears in a different form in /16,17/).

Theorem 5.2. If the conditions of Theorem 3.2 are satisfied, then: 1) the invariant set $A = O(\theta) \subset X$ of the unperturbed set and independent of p exists for all fairly small $\|p\|$; 2) for any arbitrarily small neighborhood $D = O^{\delta} \subset A$ there exists a number $\eta > 0$ and neighborhood $B = O(\theta) \subset D$ such that for all $\|p\| < \eta$ the neighborhood B is an asymptotically stable invariant set with a region of asymptotic stability that contains A.

Regions bounded by K_0'' and K_0' may be taken as the regions A and B, respectively, taking into account that $K_p \equiv K_0$ and assuming that $\|p\|$ is so small that in the region between K_0' and K_0'' $V_p^{\cdot}(x) < 0$.

Example 1. Let us consider the two-dimensional dynamic system investigated in /17/, setting in it $g = -1$ and taking μ as the parameter. We obtain the system $x_1^{\cdot} = \mu x_1^3$; $x_2^{\cdot} = -x_2$. For the Liapunov function $V = x_1^2 + x_2^2$ /17/ function $V^{\cdot}$ is negative definite when $\mu = 0$, while for $\mu > 0$ it assumes positive values in any neighborhood of zero but remains negative outside the circle $x_1^2 + x_2^2 \leqslant \mu$ which is bounded by the component of level $V = \mu$ of function V. Hence the unperturbed system ($\mu = 0$) is parametrically stable, and the circle of radius $\sqrt{\delta}$ is an asymptotically invariant set of the perturbed system for all μ smaller that some $\delta > 0$.

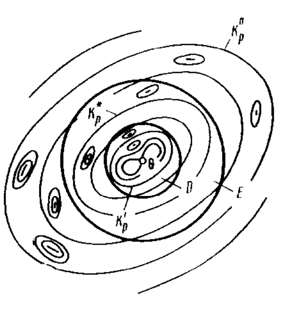

Fig.2

As shown in /22/, the theory of equilibrium bifurcation /2/ may, under specific conditions, be extended to problems of stability of steady motions of systems with ignorable coordinates by taking vector p of generalized momenta of ignorable coordinates as the parameter of steady motion surface B. It follows from Theorem 2.1 that when the Routh potential $W(x, p)$ (x is the vector of position coordinates and velocities) is positive definite with respect to x with p fixed, the respective steady motion is parametrically stable, i.e. stable not only under perturbations of x but also of p. This was first stated by Liapunov /23/; various proofs of this statement, based on a reasoning different from the one used here, appeared in /24–27/.

Establishment of the positive definiteness of the Routh potential at the bifurcation point presents difficulties owing to the degeneration there of the second differential, however, if function $W(x, p)$ is smooth with respect to x and continuous with respect to p, Lemma 2.1 implies the following statement /28/.

Theorem 5.3. If (x°, p°) is a bifurcation point and there exists a cross section $p = p(\alpha)$ (α is a scalar and $p^\circ = p(\alpha^\circ)$) of surface B, passing through it, and at which for $\alpha \leqslant \alpha^\circ$ a unique branch (right-hand branching) originates, then the positive definiteness of W with respect to x along that branch, when $\alpha < \alpha^\circ$, is sufficient for the parametric stability of stationary motion (x°, p°).

Example 2. Consider the problem of stability of permanent vertical rotations of the Lagrange spinning top (see /27/). Points lying on the straight line $\beta_2 = \beta_3, \theta = 0$ in the space $(\beta_2, \beta_3, \theta)$ (β_2 and β_3 are the generalized momenta of ignorable coordinates ψ, φ, , and θ, ψ, and φ are Euler's angles) correspond to these. When the Maievskii inequality is satisfied, the above straight line represents the unique branch of the steady motion surface at cross section $\beta_2 = \beta_3$ /27/, hence the permanent rotation is parametrically stable (*).

Let us consider a conservative system. It was shown in /30/ that, if the potential energy $\Pi(q)$ ($\Pi(0) = 0$) is continuous and has in any neighborhood of zero of the configuration space a centrally separating component of the positive level (condition A), then the total energy in the phase space has the same property, hence the equilibrium $q = 0, q^\cdot = 0$ is stable. From the proof of Theorem 5.1 follows the sufficiency of condition A for parametric stability.

Theorem 5.4. If the energy of a conservative system continuously depends on parameter p and when $p = 0$ its potential energy satisfies condition A, then the equilibrium position of the unperturbed ($p = 0$) system is parametrically stable, and when $\|p\|$ is fairly small there exists a stable invariant set that contains the phase space zero and contracts to it as $p \to 0$.

Example 3. Let a conservative system in unperturbed state have potential energy of the type of the Painlevé type (see /31/): $\Pi = \exp(-\|q\|^{-1}) \sin(\|q\|^{-1})$. The components of its level are hyperspheres that tend to point $q = 0$, and at any point of that neighborhood Π assumes positive and negative values. By virtue of Theorem 5.4 the equilibrium $q = 0, q^\cdot = 0$ is parametrically stable.

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*) As mentioned by V. V. Rumiantsev, the stability of the Lagrange top vertical rotations was first demonstrated by Chetaev /29/, where a sheaf of first integrals W is constructed and it is pointed out that when $W_2 = 0$, $W_4 > 0$. A similar result was reported by V.V Irtegov at the V. V. Rumiantsev's seminar in the Moscow State University on March 5,1976. In a paper cited in /28/ he had also investigated the stability of degenerate permanent rotations of an asymmetric solid body.

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